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Let I be the incenter of triangle ABC and let R be the circumradius. Prove that

$$AI + BI + CI \le 3R.$$

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Let r and s be, respectively, inradius and semiperimeter of $\triangle ABC$.

Noting that $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ and $AI = \dfrac{r}{\sin(A/2)} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$

we obtain $\sum AI \le 3R \Leftrightarrow \sum \sin\frac{B}{2}\sin\frac{C}{2} \le \frac{3}{4}$ and since $3\sum \sin\frac{B}{2}\sin\frac{C}{2} \le \left(\sum \sin\frac{A}{2}\right)^2$

remains to prove inequality $\left(\sum \sin\frac{A}{2}\right)^2 \le \frac{9}{4} \Leftrightarrow \left(\sum \sin\frac{A}{2}\right)^2 \le \frac{9}{4} \Leftrightarrow$

(1) $\quad \sum \sin\frac{A}{2} \le \frac{3}{2}.$

Since* $\sin\frac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ and by AM-GM Inequality

$\sqrt{\dfrac{(s-b)(s-c)}{bc}} \le \frac{1}{2}\left(\frac{s-b}{c} + \frac{s-c}{b}\right)$ we obtain $\sum \sin\frac{A}{2} \le \frac{1}{2}\sum\left(\frac{s-b}{c} + \frac{s-c}{b}\right) =$

$\frac{1}{2}\left(\sum \frac{s-b}{c} + \sum \frac{s-c}{b}\right) = \frac{1}{2}\left(\sum \frac{s-b}{c} + \sum \frac{s-a}{c}\right) = \frac{1}{2}\sum \frac{2s-a-b}{c} = \frac{1}{2}\sum \frac{c}{c} = \frac{3}{2}.$

* $a^2 = b^2 + c^2 - 2bc\cos A \Leftrightarrow a^2 - (b-c)^2 = 2bc(1-\cos A) \Leftrightarrow$

$(a-b+c)(a+b-c) = 4bc\sin^2\frac{A}{2} \Leftrightarrow \sin\frac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$